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Examples of Orthomorphisms

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1. INTRODUCTION

In the present paper L will denote a *Riesz space* (vector lattice), that is to say, L is a real vector space which is at the same time a lattice such that the vector space structure and the lattice structure are compatible (this means that if $f \leqslant g$ holds in L, then $f + h \leqslant g + h$ for all $h \in L$ and $af \leqslant ag$ for all real $a \geqslant 0$). The least upper bound and the greatest lower bound of f and g in L will be denoted by $\sup(f, g)$ and $\inf(f, g)$, respectively; the customary abbreviations $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$ will be used, and the absolute value $|f|$ of f is defined by $|f| = f^+ + f^-$. Elements $f \geqslant 0$ are called *positive elements*; the set L^+ of all positive elements is the *positive cone*, and elements f, g satisfying $\inf(|f|, |g|) = 0$ are said to be *disjoint* (notation: $f \perp g$).

A familiar example is the vector space $C([a, b])$ of all real continuous functions on the closed interval $[a, b]$. To say that $f \leqslant g$ holds in this space simply means that $f(x) \leqslant g(x)$ for all $x \in [a, b]$, and $\sup(f, g)$ and $\inf(f, g)$ are the ordinary pointwise supremum and infimum. Another example is the space $C_c(-\infty, \infty)$ of all real continuous functions on $(-\infty, \infty)$ possessing a compact carrier. An example of a somewhat different nature is obtained by considering a measure space (X, Λ, μ) and a number q $(0 < q < \infty)$, and taking now the vector space $L^q = L^q(X, \mu)$ of all real μ-measurable functions f on X having the property that the integral of $|f|^q$ over X is finite. In this example functions differing only on a set of measure zero have to be identified, and $f \leqslant g$ means now that $f(x) \leqslant g(x)$ holds for μ-almost every $x \in X$.

Given the Riesz space L, a linear mapping π from L into itself is called a *positive orthomorphism* whenever $\inf(f, g) = 0$ implies $\inf(\pi f, g) = 0$. The general theory of orthomorphisms has been developed in some recent papers [2, 4], but these papers do not contain many examples. In particular, one would like to know all possible orthomorphisms in the spaces mentioned above. In the space $C([a, b])$ it is easy enough to indicate a certain class of positive orthomorphisms. Indeed, given a nonnegative function $p(x)$ in $C([a, b])$, the linear mapping π in $C([a, b])$, defined by $(\pi f)(x) = p(x) \cdot f(x)$, is evidently a positive orthomorphism. The question arises whether every positive orthomorphism in $C([a, b])$ is of this kind. We shall prove in the present paper that the answer is affirmative. It is of course not difficult to guess that this has something to do with the ring structure of $C([a, b])$. In some other examples, however, we shall have to step beyond the ring structure. In the space $C_c(-\infty, \infty)$ for example one sees immediately that if $p(x)$ is a nonnegative continuous function on $(-\infty, \infty)$, not necessarily with a compact carrier, then pointwise multiplication by $p(x)$ is a positive orthomorphism. We shall prove that every positive orthomorphism in $C_c(-\infty, \infty)$ is of this kind. Finally, in the example of the space L^q it is evident that pointwise multiplication by a nonnegative L^∞ function is a positive orthomorphism, and also for this example it will be proved that every positive orthomorphism is such a multiplication.

In the proofs we shall need an interesting theorem about commutativity of so-called *f-algebras*. We explain what an f-algebra is. The Riesz space L is called a Riesz algebra if there exists in L a (not necessarily commutative) ring multiplication with the usual algebra properties (in particular $(af)\,g = f(ag) = a(fg)$ for real a) and such that $f, g \in L^+$ implies $fg \in L^+$. The Riesz algebra is then called an f-algebra if $\inf(f, g) = 0$ implies $\inf(hf, g) = \inf(fh, g) = 0$ for every $h \in L^+$; in other words, whenever left and right multiplication by positive elements are positive orthomorphisms. The notion of an f-algebra is derived from the example that L consists of real functions on a point set X with the Riesz space structure defined pointwise. If in this case the ordinary pointwise product fg is a member of L for all f and g in L, then L is automatically an f-algebra, and L is then even a commutative f-algebra. The theorem referred to above is the theorem that every Archimedan f-algebra is commutative (the Riesz space L is called *Archimedean* if it follows from $0 \leqslant nf \leqslant g$ for $n = 1, 2,\ldots$ that $f = 0$). This commutativity result was proved first by G. Birkhoff and R. S. Pierce [3]; their proof is based on what they call a "metamathematical theorem," namely, the theorem that the class of Riesz algebras is "equationally definable." Due to this the Archimedan f-algebra L can be treated in the proof as if L were linearly ordered; it follows then that

$$n\,|fg - gf| \leqslant f^2 + g^2,$$

holds for all $f, g \in L^+$ and all natural numbers n. Since L is Archimedean, $fg = gf$ is an immediate consequence. Several years later S. J. Bernau [1] published an elementary proof for the above inequality. We present a third commutativity proof; a very simple proof although it cannot be called elementary because it depends ultimately upon some nontrivial facts about orthomorphisms.

Given the Riesz space L and the non-negative number a, it is evident that multiplication by a is a positive orthomorphism in L. Let us call this a *trivial orthomorphism*. It may be asked whether there exist Riesz spaces possessing only trivial orthomorphisms. The answer is affirmative; we shall prove that the Riesz space of all real continuous functions f on $[0, 1]$ that are piecewise linear (i.e., the graph of f consists of a finite number of line segments) is an example. This example arose from some remarks made at a lecture given at the Mathematical Institute of Nijmegen University.

2. Orthomorphisms in Archimedean f-algebras

We first collect some simple facts about Riesz spaces. The linear subspace K of the Riesz space L is called a Riesz subspace of L if $f, g \in K$ implies $\sup(f, g) \in K$ and $\inf(f, g) \in K$. The subset D of L is said to be *solid* if it follows from $f \in D$ and $|g| \leq |f|$ that $g \in D$. Any solid linear subspace is called an *ideal*. Every ideal is a Riesz subspace. Any ideal I with the extra property that whenever a subset of I has a least upper bound in L then that least upper bound is a member of I, is called a *band*. Given the (nonempty) subset D of L, the set D^d of all $g \in L$ satisfying $g \perp f$ for all $f \in D$ is called the *disjoint complement* of D. The disjoint complement of D^d will be denoted by D^{dd}. It is evident from the definition that D is a subset of D^{dd}. Hence, since $D_1 \subset D_2$ obviously implies $D_1{}^d \supset D_2{}^d$ it follows from $D \subset D^{dd}$ that $D^d \supset D^{ddd}$. On the other hand, replacing D by D^d in $D \subset D^{dd}$, we obtain $D^d \subset D^{ddd}$. Hence $D^d = D^{ddd}$. In other words, if B is the disjoint complement of some nonempty set, then $B = B^{dd}$. We shall also need the notion of order denseness. The subset D of L is called *order dense* if $D^d = \{0\}$ or, equivalently, if $D^{dd} = L$.

Now, let π be a positive orthomorphism in the Riesz space L, i.e., $\inf(f, g) = 0$ implies $\inf(\pi f, g) = 0$. It follows that $\inf(f, g) = 0$ also implies $\inf(\pi f, \pi g) = 0$, and it is easy to derive from this that

$$\pi\{\inf(f, g)\} = \inf(\pi f, \pi g),$$

holds for all $f, g \in L$. The same formula with inf replaced by sup is an immediate consequence. Hence, the mapping π preserves finite suprema and infima. Any linear mapping in L preserving finite suprema and infima is called a *Riesz homomorphism*. Any positive orthomorphism is, therefore, a Riesz homomorphism. The converse is not true; if $L = C(-\infty, \infty)$ and π is a translation, i.e., $(\pi f)(x) = f(x + a)$ for some constant $a \neq 0$, then π is a Riesz homomorphism, but not an orthomorphism. Since any Riesz homomorphism π preserves finite suprema and infima, the range R_π of π is a Riesz subspace of L. This holds in particular, therefore, if π is a positive orthomorphism. In some cases, for example if π is pointwise multiplication by a strictly positive function in the space $C([a, b])$, the range R_π is an ideal. There exist also examples, however, that R_π is not an ideal in L. Let L be the Riesz space consisting of all real continuous functions f on $[0, 1]$ such that $f(0) = 0$ and the (right) derivative $f'(0)$ exists as a finite number. The positive orthomorphism π in L is defined by $(\pi f)(x) = x \cdot f(x)$. Then the function $f(x) = x^2$ is in R_π and $g(x) = x^2 \sin x^{-1} \in L$ with $|g| \leqslant f$. But g is not a member of R_π because otherwise we should have $x \sin x^{-1} \in L$. Hence, R_π is not an ideal. The kernel (null space) K_π and the range R_π of a positive orthomorphism π in the Archimedean Riesz space L are related by the formula $K_\pi = (R_\pi)^d$; the proof is not trivial but elementary.

Any linear mapping π in the Riesz space L such that $\pi = \pi_1 - \pi_2$ with π_1 and π_2 positive orthomorphisms is called an *orthomorphism*. One of the fundamental results about orthomorphisms in an Archimedean Riesz space is that the formula $K_\pi = (R_\pi)^d$ continues to hold in this more general case. In [2] and [4], this was proved by means of the representation theory for Archimedean Riesz spaces. It is shown then that for any Archimedean Riesz space L there exists an isomorphic space $L^\wedge$ consisting of real continuous (or continuous in the extended sense that $+\infty$ and $-\infty$ are allowed as values) functions on a topological space Ω; an orthomorphism π in L corresponds with pointwise multiplication by an appropriate function $p(\omega)$ in $L^\wedge$. The space Ω, although compact and Hausdorff, is extremally disconnected (i.e., the closure of every open set is open). Thus, if for example L is the space $C([0, 1])$, the corresponding space $L^\wedge$, although consisting again of continuous functions, cannot be identified with the original space L in the sense that Ω and $[0, 1]$ can be regarded as identical, because $[0, 1]$ is not extremally disconnected. In this respect, therefore, the situation is different from the situation for the Gelfand representation, where $C([0, 1])$ is regarded as a commutative Banach algebra. For this reason, although every orthomorphism in $L^\wedge$ is a pointwise multiplication, we may not yet conclude that the same holds in $L = C([0, 1])$.

We prove now that if π_1 and π_2 are orthomorphisms coinciding on an order dense set in the Archimedean Riesz space L, then $\pi_1 = \pi_2$.

THEOREM 1. *Let π_1 and π_2 be orthomorphisms in the Archimedean Riesz space L such that π_1 and π_2 coincide on the order dense set D. Then $\pi_1 = \pi_2$.*

Proof. The mapping $\pi = \pi_1 - \pi_2$ is also an orthomorphism in L. It follows from $K_\pi = (R_\pi)^d$ that K_π is a disjoint complement, so $(K_\pi)^{dd} = K_\pi$ by one of our earlier remarks. Since $D \subset K_\pi$ holds by hypothesis and since $D^{dd} = L$ (because D is order dense), we have

$$L = D^{dd} \subset (K_\pi)^{dd} = K_\pi ,$$

so $K_\pi = L$. This shows that π is the null mapping, and so $\pi_1 = \pi_2$.

Assume now that f and g are elements in the f-algebra L satisfying $\inf(f, g) = 0$. Then f and g are positive elements, so right multiplication by g is a positive orthomorphism, i.e., $\inf(fg, g) = 0$, so $fg \in \{g\}^d$. Similarly it follows from $\inf(g, h) = 0$ for any $h \geq 0$ in $\{g\}^d$ that $\inf(fg, h) = 0$, so $fg \in \{g\}^{dd}$. Hence $fg \in \{g\}^d \cap \{g\}^{dd}$, so $fg = 0$. Now, let $p \perp q$ in L. Then

$$\inf(p^+, q^+) = \inf(p^+, q^-) = \inf(p^-, q^+) = \inf(p^-, q^-) = 0,$$

so $p^+q^+ = p^+q^- = p^-q^+ = p^-q^- = 0$ by what was proved already. It follows that $pq = 0$. It has been proved thus that in an f-algebra $p \perp q$ implies $pq = 0$. We use this in the proof of the commutativity theorem.

THEOREM 2. *Every Archimedean f-algebra is commutative.*

Proof. Given the Archimedean f-algebra L, let p be an arbitrary element of L^+. Multiplication on the left by p is a positive orthomorphism π_l in L and multiplication on the right by p is a positive orthomorphism π_r in L. For each f satisfying $f \perp p$ we have $pf = fp = 0$, so π_l and π_r coincide on $\{p\}^d$. Also, $\pi_l p = p^2 = \pi_r p$, so π_l and π_r coincide for p. Now, let D be the subset of L consisting of p and $\{p\}^d$. Then $D^d = \{0\}$, so D is order dense. Furthermore, π_l and π_r coincide on D. Hence, by the preceding theorem, we have $\pi_l = \pi_r$, i.e., $pf = fp$ for all $f \in L$. It follows easily that multiplication is commutative.

In the next theorem it will be proved that in an Archimedean f-algebra with unit element every orthomorphism is a multiplication.

THEOREM 3. *Let L be an Archimedean f-algebra with unit element (with respect to multiplication). Given the orthomorphism π in L, there exists an element $p \in L$ such that $\pi f = pf$ holds for every $f \in L$. Conversely, every $p \in L$ gives rise to an orthomorphism π in L defined by $\pi f = pf$ for all $f \in L$. The orthomorphism π is positive if and only if the corresponding element p is positive.*

Proof. Let e be the unit element of L. Given the orthomorphism π in L, let $p = \pi e$. The product pf is in L for every $f \in L$, so $\pi_1 f = pf$ defines a linear mapping π_1 from L into itself. Evidently, multiplication by p^+ is a positive orthomorphism in L (by the definition of an f-algebra), and the same holds for multiplication by p^-. Hence, multiplication by p is an orthomorphism in L, that is to say, π_1 is an orthomorphism in L. The orthomorphisms π and π_1 coincide on the set $\{e\}$, consisting of e only. The set $\{e\}$ is order dense in L (indeed, if $f \in \{e\}^d$, then $f \perp e$, so $fe = 0$ by one of the remarks made earlier, i.e., $f = 0$). It follows then from Theorem 1 that $\pi = \pi_1$, so $\pi f = pf$ for all $f \in L$.

EXAMPLES. (i) Let the Riesz space L consist of real functions on a nonempty point set X (with the vector space structure and the order structure in the usual pointwise manner), and in addition let L be an algebra with respect to pointwise multiplication. Furthermore, let the unit function e, satisfying $e(x) = 1$ for all $x \in X$, be a member of L. Then L is an Archimedean f-algebra with unit. Hence, the linear mapping π in L is an orthomorphism if and only if there exists a function $p \in L$ such that $(\pi f)(x) = p(x) \cdot f(x)$ holds on X for every $f \in L$. This holds in particular if L is the space $C(X)$ of all real continuous functions on a topological space X or the space $C_b(X)$ of all bounded real continuous functions on X.

(ii) Let L be the space $M(X, \mu)$ of all real and μ-almost everywhere finite μ-measurable functions on the point set X, where μ is a measure in X and where μ-almost equal functions are identified. The orthomorphisms in L are the pointwise multiplications. The same holds if L is the subspace $L^\infty(X, \mu)$ of all bounded functions in $M(X, \mu)$.

(iii) The theorem covers the case that L is one of the sequence spaces (s), l^∞ or (c). We recall that (s) is the space of all real sequences and (c) is the space of all convergent real sequences. The theorem can also be applied if L is the space of all real sequences $f = (f_1, f_2, \ldots)$ with f_n constant for $n \geq n_0$ (with n_0 depending upon f). The same holds if L is the space of all real sequences $f = (f_1, f_2, \ldots)$ such that f_n assumes only finitely many different values (these values depending upon f).

(iv) Given the Hilbert space H (over the complex numbers), let $\mathscr{H}$ be the ordered vector space of all bounded Hermitian operators in H (if $A, B \in \mathscr{H}$, then $A \leq B$ whenever $(Ax, x) \leq (Bx, x)$ for all $x \in H$). The space $\mathscr{H}$ is not a Riesz space, unless in the trivial case that H is one-dimensional. Let $\mathscr{D}$ be a nonempty subset of $\mathscr{H}$ such that all members of $\mathscr{D}$ commute mutually, and let $\mathscr{C}''(\mathscr{D})$ be the second commutant of $\mathscr{D}$. Then $\mathscr{C}''(\mathscr{D})$ is an Archimedean Riesz ring, and the members of $\mathscr{C}''(\mathscr{D})$ commute mutually (c.f. [6, Section 55]). Furthermore, for $A, B \in \mathscr{C}''(\mathscr{D})$, we have $A \perp B$ if and only if $AB = \theta$, where θ is the null operator. It follows that $\mathscr{C}''(\mathscr{D})$ is an Archimedean f-algebra. Hence, the mapping π from $\mathscr{C}''(\mathscr{D})$ into itself is an orthomorphism if and only if there exists $B \in \mathscr{C}''(\mathscr{D})$ such that $\pi(A) = BA$ holds for all $A \in \mathscr{C}''(\mathscr{D})$.

The case that L is the space of all real sequences with only finitely many non-zero terms is not covered by Theorem 3 because the unit sequence is not in L. The following theorem takes care of this case and similar cases.

THEOREM 4. *Let the Riesz space L consist of real functions on the point set X, where X is the disjoint union of the collection of subsets $(X_\alpha\colon \alpha \in \{\alpha\})$. Let e_α be the characteristic function of X_α, and let it be given that for each $\alpha \in \{\alpha\}$ and for all $f, g \in L$ the functions e_α, fe_α, ge_α and fge_α are members of L. Given the orthomorphism π in L, we set $p_\alpha = \pi e_\alpha$ for each α. Then p_α vanishes outside X_α, and denoting by p the function on X equal to p_α on each X_α, we have $(\pi f)(x) = p(x) \cdot f(x)$ for all $f \in L$.*

Proof. There is no less of generality if we assume that π is a positive orthomorphism. It follows then from $\inf(e_\alpha, e_\beta) = 0$ for $\alpha \neq \beta$ that $\inf(\pi e_\alpha, e_\beta) = 0$, so $p_\alpha = \pi e_\alpha$ vanishes outside X_α. Note now that $f \perp e_\alpha$ implies $\inf(f^+, e_\alpha) = \inf(f^-, e_\alpha) = 0$, so $\inf(\pi f^+, e_\alpha) = \inf(\pi f^-, e_\alpha) = 0$, i.e., $\pi f \perp e_\alpha$. In other words, if $f \in L$ vanishes on X_α, then πf vanishes on X_α. It follows that if f_1 and f_2 coincide on X_α, then πf_1 and πf_2 conicide on X_α. Given an arbitrary $f \in L$, the functions f and $f_\alpha = fe_\alpha$ coincide on X_α, so πf and πf_α coincide on X_α, i.e.,

$$(\pi f)(x) = (\pi f_\alpha)(x) \qquad \text{for} \quad x \in X_\alpha .$$

The functions of L, with domain restricted to X_α, form an Archimedean f-algebra with unit e_α, so

$$(\pi f_\alpha)(x) = p_\alpha(x) \cdot f_\alpha(x) = p(x) \cdot f(x), \qquad \text{for} \quad x \in X_\alpha .$$

Combining the so obtained results, we get

$$(\pi f)(x) = p(x) \cdot f(x), \qquad \text{for} \quad x \in X_\alpha .$$

The last formula holds for all $\alpha \in \{\alpha\}$ simultaneously, and so $(\pi f)(x) = p(x) \cdot f(x)$ holds for all $x \in X$.

EXAMPLES. (v) If π is an orthomorphism in the space of all real sequences $f = (f_1, f_2, \ldots)$ with only finitely many nonzero terms, then there exists a real sequence $p = (p_1, p_2, \ldots)$ such that $(\pi f)_n = p_n f_n$ for all n. Conversely, every real sequence thus defines an orthomorphism in this space.

(vi) If π is an orthomorphism in the space (c_0) of all real sequences converging to zero, then there exists a real sequence $p = (p_1, p_2, \ldots)$ such that $(\pi f)_n = p_n f_n$ for all n. It is easy to see that in addition the sequence $(p_1, p_2, \ldots)$ must be bounded in order that $\pi f \in (c_0)$ should hold for every $f \in (c_0)$. Conversely, every bounded sequence thus defines an orthomorphism in (c_0).

3. Orthomorphisms in the Space of Real Continuous Functions on a Locally Compact Hausdorff Space

In the present section we assume that X is a locally compact Hausdorff space and L is the Riesz space of all real continuous functions on X with compact carrier, i.e., $L = C_c(X)$.

Theorem 5. *If π is an orthomorphism in $L = C_c(X)$, there exists a real continuous function $p(x)$ on X such that $(\pi f)(x) = p(x) \cdot f(x)$ holds for all $f \in L$. Conversely, every real continuous $p(x)$ on X gives rise to an orthomorphism in L.*

Proof. (i) Given the open subset A of X with compact closure A^-, there exists an open subset B with compact closure such that $A^- \subset B$. Indeed, assign to each $x \in A^-$ an open neighborhood U_x with compact closure. Then $\bigcup (U_x : x \in A^-)$ is an open covering of A^-, so there exists a finite subcovering. The union of the sets in the finite subcovering may be chosen as the required set B. It follows that there exists a real continuous function $e_A(x)$ on X such that $0 \leqslant e_A(x) \leqslant 1$ on X, $e_A(x) = 1$ for $x \in A$ and $e_A(x) = 0$ outside B. Note that e_A is, therefore, a member of L.

(ii) Let π be an orthomorphism in L. It is no restriction for our purposes to assume that π is a positive orthomorphism. We show first that if $f \in L$ and f vanishes on the open subset A of X, then πf vanishes on A. We may assume that $f \in L^+$. Given any point $x_0 \in A$, there exists a function $u_0(x) \in L$ such that $0 \leqslant u_0(x) \leqslant 1$ on X, $u_0(x_0) = 1$ and $u_0(x) = 0$ outside A. Then $\inf(f, u_0) = 0$, so $\inf(\pi f, u_0) = 0$, which implies that $(\pi f)(x_0) = 0$. This holds for every $x_0 \in A$, so πf vanishes on A.

(iii) Let $(A_\alpha : \alpha \in \{\alpha\})$ be the collection of all open subsets of X such that A_α^- is compact. The collection is directed upwards with respect to partial ordering by inclusion. To each α we assign a function $e_\alpha \in L$ satisfying $0 \leqslant e_\alpha(x) \leqslant 1$ on X and $e_\alpha(x) = 1$ on A_α (this is possible by (i)). Furthermore, if π is the given positive orthomorphism, let $\pi e_\alpha = p_\alpha$ for every $\alpha \in \{\alpha\}$. Note that for a point x_α in A_α the value $p_\alpha(x_\alpha)$ does not depend on the values assumed by e_α outside A_α (this follows from (ii)). Note also that $p_\alpha(x) = p_\beta(x)$ for $x \in A_\alpha \cap A_\beta$ (by (ii) again). Hence, this common value may be denoted simply by $p(x)$. Since each $x \in X$ is a point of at least one A_α, the function $p(x)$ is defined on the whole set X. Evidently $p(x)$ is continuous and nonnegative on X.

(iv) Let $u(x)$ be a nonnegative function in L, and denote the set $(x : u(x) > 0)$ by A. It will be proved that $(\pi u)(x) = p(x) \cdot u(x)$ holds for all $x \in X$, and it is sufficient, therefore, to show this for all $x \in A$ (since $u(x) = 0$ on the complement of A^- implies $(\pi u)(x) = 0$ on that set). The set of all $f \in L$, with domain restricted to A, is a Riesz space L' of real continuous functions on A with the characteristic function χ_A of A as one of its members (note that A is one of the A_α, so χ_A is the restriction of the corresponding e_α). The space L' is a ring with respect to pointwise multiplication. The function $p(x)$ is on $A = A_\alpha$ the restriction of the corresponding $p_\alpha(x)$, so $pf \in L'$ for all $f \in L'$. This shows that multiplication by $p(x)$ gives rise to an orthomorphism in L'. The "restriction" to L' of the given orthomorphism π is an orthomorphism π' in L', given by

$$\pi'(f\chi_A) = (\pi f) \cdot \chi_A, \qquad \text{for all} \quad f \in L.$$

The mapping π' is well defined on L' because $f\chi_A = g\chi_A$ implies that πf and πg coincide on A (by (ii)). Note that $\pi' u = \pi u$ on A for the given function $u \in L^+$, and note also that $\pi'(\chi_A) = p \cdot \chi_A$, so π' coincides with "multiplication by p" on the subset of L' consisting of χ_A alone. This subset is order dense in L', so π' coincides with multiplication by p on L'; in particular $\pi' u = p \cdot u$ on A. It follows that $(\pi u)(x) = p(x) \cdot u(x)$ for all $x \in A$. As observed above, the same holds now for all $x \in X$. This is true now for every $u \in L^+$, so it is an immediate consequence that $(\pi f)(x) = p(x) \cdot f(x)$ holds on X for every $f \in L$.

4. Spaces with Only Trivial Orthomorphisms

As observed in the introduction, every orthomorphism π in the Riesz space L that is defined by $\pi f = af$ for some real constant a is called a trivial orthomorphism. We prove that there exist Archimedean Riesz spaces possessing only trivial orthomorphisms.

Theorem 6. *Let L be the Riesz space of all real continuous functions f on $[a, b]$ such that f is piecewise linear (i.e., the graph of f consists of a finite number of line segments). Then L has only trivial orthomorphisms.*

Proof. L is a Riesz subspace of $C([a, b])$. We show that every orthomorphism in L can be extended to an orthomorphism in $C([a, b])$. Note first that if $f \in C([a, b])$ and $\epsilon > 0$ are given, there exist $s, t \in L$ such that $s \leqslant f \leqslant t$ and $t(x) - s(x) \leqslant \epsilon$ for all $x \in [a, b]$. Indeed, consider a partition of $[a, b]$ such that the oscillation of f on each subinterval $[x_{i-1}, x_i]$ is at most $\epsilon/3$, then let f^* be the continuous function such that $f^*(x_i) = f(x_i)$ at all points x_i and such that f^* is linear on each $[x_{i-1}, x_i]$, and finally let $s = f^* - \epsilon/3$ and $t = f^* + \epsilon/3$. Now let $\epsilon_n \downarrow 0$ and denote the corresponding s and t by s_n and t_n. We may assume that $s_n \uparrow$ and $t_n \downarrow$ (otherwise, replace s_n by $\sup(s_1, \ldots, s_n)$).

Let $e(x) = 1$ for all $x \in [a, b]$ and let π be a positive orthomorphism in L. The function e is a member of L, so πe makes sense. Let $(\pi e)(x) = p(x)$ on $[a, b]$, so $0 \leqslant p \in L$. It follows from $0 \leqslant s_m(x) - s_n(x) \leqslant e_n$ for $m \geqslant n$ that

$$0 \leqslant \pi s_m - \pi s_n \leqslant \epsilon_n p,$$

so the sequence $(\pi s_n: n = 1, 2, \ldots)$ of continuous functions converges uniformly on $[a, b]$. The limit function $g(x)$ is therefore continuous. Evidently g does not depend on the particular approximating sequence $(s_n: n = 1, 2, \ldots)$. Indeed, it is easy to see that

$$g = \sup(\pi s: s \in L, s \leqslant f).$$

Set $g = \pi f$. This can be done for every $f \in C([a, b])$. It is evident that π is a positive orthomorphism in $C([a, b])$. Hence, by one of our earlier results, $(\pi f)(x) = p(x) \cdot f(x)$ holds for every $f \in C([a, b])$; in particular this holds for every $f \in L$. It follows now from $p = \pi e \in L$ that p is a piecewise linear function. The product $p \cdot f$ must likewise be piecewise linear for every $f \in L$. Also, p must be continuous. This is possible only if p is a nonnegative constant. Hence, π is trivial.

If the natural number n is fixed and L is the Riesz space of all real continuous functions f on $[a, b]$ such that f is piecewise a polynomial of degree at most n, then again L has only trivial orthomorphisms. On the other hand, if L is the space of all real continuous functions f such that f is piecewise a polynomial (without restrictions on the degree), then L is an Archimedean f-algebra, and so (by Theorem 3) every orthomorphism is a pointwise multiplication by a function of L.

5. Orthomorphisms in L^q and in Banach Function Spaces

For the determination of all orthomorphisms in an L^q space we need a few more facts about orthomorphisms in an arbitrary Archimedean Riesz space L. The collection Orth(L) of all orthomorphisms in L is obviously a real vector space, partially ordered by defining that $\pi_1 \geqslant \pi_2$ holds whenever $\pi_1 - \pi_2$ is a positive orthomorphism. It can be proved now that Orth(L) is a Riesz space; given π_1 and π_2 in Orth(L), the greatest lower bound $\pi_1 \wedge \pi_2$ in Orth(L) exists and satisfies

$$(\pi_1 \wedge \pi_2)(f) = \inf(\pi_1(f), \pi_2(f)) \qquad \text{for all} \quad f \in L^+.$$

Denoting by I the identity transformation in L, it follows in particular that, for any positive orthomorphism π and any natural number n, the orthomorphism $\pi_n = \pi \wedge nI$ exists and π_n satisfies

$$\pi_n f = (\pi \wedge nI)(f) = \inf(\pi(f), nf) \qquad \text{for all} \quad f \in L^+. \tag{1}$$

Now, let $u, v \in L^+$ be such that u is a member of the band generated by v. It is well-known that in this case $\inf(u, nv) \uparrow u$ as $n \to \infty$. In other words, u is the supremum of the sequence $(\inf(u, nv): n = 1, 2, \ldots)$. We apply this to the situation in formula (1). Here the element $\pi(f)$ is in the band generated by f, and hence we have

$$\pi_n f = \inf(\pi f, nf) \uparrow \pi f \quad \text{as} \quad n \to \infty \qquad \text{for all} \quad f \in L^+. \tag{2}$$

In the case that L is the space L^q $(0 < q < \infty)$ on a measure space of σ-finite measure, the supremum in L^q of a sequence is just the pointwise supremum (μ-almost everywhere).

THEOREM 7. *Let μ be a (totally) σ-finite measure in the point set X, let $0 < q < \infty$ and let L be the Riesz space $L^q = L^q(X, \mu)$. Then, given the orthomorphism π in $L = L^q$, there exists a real function $p(x) \in L^\infty$ such that $(\pi f)(x) = p(x) \cdot f(x)$ holds on X for every $f \in L^q$. Conversely, every $p \in L^\infty$ gives thus rise to an orthomorphism π in L^q.*

Proof. We assume first that $\mu(X)$ is finite. In this case the unit function e, satisfying $e(x) = 1$ for all $x \in X$, is an element of L^q. Note that Theorem 3 is not applicable, because in general L^q is not an algebra. We proceed, therefore, somewhat differently.

Given the positive orthomorphism π in L^q, let $\pi e = p$. Then $0 \leqslant p(x) \in L^q$. For $n = 1, 2, \ldots$, let $\pi_n = \pi \wedge nI$, and let $p_n(x) = \min(p(x), n)$ for all $x \in X$. Since $p_n \cdot f \in L^q$ for every $f \in L^q$, it is evident that the mapping π_n', defined by

$$(\pi_n' f)(x) = p_n(x) \cdot f(x), \qquad \text{for all} \quad f \in L^q,$$

is a positive orthomorphism in L^q. Observe now that

$$\pi_n e = \inf(\pi e, ne) = \inf(p, ne) = p_n = \pi_n' e,$$

so π_n and π_n' coincide on the order dense set $\{e\}$ and hence $\pi_n = \pi_n'$. In other words, we have

$$(\pi_n f)(x) = p_n(x) \cdot f(x), \qquad \text{for all} \quad f \in L^q.$$

Now assume that $0 \leqslant f_0 \in L^q$. As observed in formula (2), we have $0 \leqslant \pi_n f_0 \uparrow \pi f_0$ as $n \to \infty$ in the space L^q, which means (as also observed already) that the function πf_0 is the pointwise supremum of the sequence of functions $(\pi_n f_0: n = 1, 2, \ldots)$. Also,

$$0 \leqslant p_n(x) \cdot f_0(x) \uparrow p(x) \cdot f_0(x)$$

holds pointwise on X. It follows that $(\pi f_0)(x) = p(x) \cdot f_0(x)$. Hence

$$(\pi f)(x) = p(x) \cdot f(x), \qquad \text{for all} \quad f \in L^q.$$

It remains to prove that $p(x) \in L^\infty$. For this purpose, let $g(x)$ be an arbitrary (real) μ-summable function on X. Then $|g|^{1/q} \in L^q$, so $p \cdot |g|^{1/q} \in L^q$ by the result just established. Hence, if g is summable, then $p^q \cdot g$ is summable. It is wellknown that in a measure space of σ-finite measure this is possible only if $p^q \in L^\infty$, i.e., if $p \in L^\infty$.

The extension to the case that X is the union of a countable number of sets of finite measure is evident, and so is the extension to the case that π is an arbitrary (not necessarily positive) orthomorphism.

Finally, let X be as above and let L be a Banach function space L_ρ (i.e., a linear subspace of the space of all real μ-measurable functions on X, normed and norm complete with respect to the Riesz norm ρ). Without loss of generality we may assume that X is saturated, i.e., X is a countable union of sets X_k ($k = 1, 2,\ldots$) such that for each k the characteristic function of X_k is an element of L_ρ. Also, since L_ρ is Banach, the Riesz-Fischer inequality holds, i.e., $\rho(\sum_1^\infty f_n) \leqslant \sum_1^\infty \rho(f_n)$ if $0 \leqslant f_n \in L_\rho$ and if the right-hand side converges.

THEOREM 8. *If L is the Banach function space L_ρ and π is an orthomorphism in $L = L_\rho$, then there exists a real function $p(x) \in L^\infty$ such that $(\pi f)(x) = p(x) \cdot f(x)$ for all $f \in L_\rho$. Conversely, every $p \in L^\infty$ gives thus rise to an orthomorphism π in L_ρ.*

Proof. Let π be a positive orthomorphism in L_ρ. The proof that there exists a function $p(x) \geqslant 0$ such that $(\pi f)(x) = p(x) \cdot f(x)$ holds for all $f \in L_\rho$ is exactly as in the L^q case. It remains to prove that $p(x)$ is bounded. This can be done by means of a variant of the method used by G. G. Lorentz and D. G. Wertheim [5] to show that f is an element in the associate space L_ρ' of L_ρ if and only if fg is summable for every $g \in L_\rho$. Assume that $p(x)$ is not bounded. Then there exists for every $n = 1, 2,\ldots$ a set E_n of finite positive measure such that $p(x) \geqslant n^3$ for all $x \in E_n$. Let $0 \leqslant f_n(x) \in L_\rho$ be such that f_n vanishes outside E_n and $\|f_n\| = \rho(f_n) = 1$. The existence of a function f_n of this kind follows from the hypothesis that X is saturated. Then $\|\pi f_n\| \geqslant n^3$ for all n. Now, let $f(x) = \sum_1^\infty n^{-2} f_n(x)$ for all $x \in X$. By the Riesz-Fischer inequality we have

$$\|f\| \leqslant \sum n^{-2} \|f_n\| = \sum n^{-2} < \infty,$$

so $f \in L_\rho$, which implies $\pi f \in L_\rho$. On the other hand it follows from $f \geqslant n^{-2} f_n \geqslant 0$, that $\pi f \geqslant n^{-2} \pi f_n \geqslant 0$, so $\|\pi f\| \geqslant n^{-2} \|\pi f_n\| \geqslant n$, for all n. This is impossible. Hence, $p(x)$ is bounded.

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